Some more practice problems for math 241

1. Find a smooth function $u(x,y)$ defined on the upper half-plane $\{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$ which satisfied the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$u(x,0) = \frac{2}{x^2+4}, \qquad \lim_{x^2+y^2 \to \infty} u(x,y) = 0.$$

(Hint: Consider the Fourier transform $U(\omega, y)$ in the x-variable of the function $u(x,y)$.)

2. Let $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \geq 1\}$, the *exterior* of a unit circle on the plane. Find a smooth and bounded function $u(x,y)$ on D such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\, r^{-4}$$

and

$$u(\cos\theta, \sin\theta) = 3\cos(3\theta) - \sin(4\theta) \quad \text{for all } \theta.$$

3. (a) Consider the eigenvalue problem,

$$\left[\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{6}{x^2} + \lambda\right] u(x), \quad u(1) = 0, \ u(x) \text{ bounded near } 0,$$

where the unknown function $u(x)$ is defined on the interval $x \in [0,1]$. Find a function $h(x)$ so that after multiply the above equation by $h(x)$, the resulting equation has the standard Sturm–Liouville form

$$(pu')' + qv + \lambda\sigma u.$$

Also, find the functions $p(x)$, $q(x)$ and $\sigma(x)$.

(b) What are the eigenvalues of this equation? (Your answer will involve certain special functions.)

4. Find the general harmonic quartic homogenous polynomial $u(x,y,z)$, i.e. $u(x,y,z)$ is homogeneous of degree 4, and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

(Please give a set of linearly independent quartic homogeneous polynomials $u_1, \ldots, u_m$, such that every quartic homogeneous polynomial u can be written as $u = c_1u_1 + \cdots + c_mu_m$ for suitable numbers $c_1, \ldots, c_m$ uniquely determined by u.)

5. Let

$$x^2 = \sum_{i=1}^{\infty} a_i\, J_0(j_i x), \quad x \in [0,1]$$

be the Fourier–Bessel expansion of the function x^2 on $[0,1]$, where $j_1 < j_2 < \cdots$ are the positive zeros of the Bessel function $J_0(x)$. Find a closed-form expression of

$$\sum_{i=1}^{\infty} a_i^2 J_1(j_i)^2.$$

(You can use the formulas $\int x^3\, J_0(x)\, dx = x(x^2-4)J_1(x) + 2x^2 J_0(x) + C$ and $\int_0^1 x\, J_0(ax)^2\, dx = \frac{1}{2}\big(J_0(a)^2 + J_1(a)^2\big) \quad \forall a \in \mathbb{R}$.

6. Find a function $u(x,y,t)$ defined on

$$\{(x,y,t) \in \mathbb{R}^3 \mid x^2+y^2 \le 1,\ t \ge 0\}$$

which satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

and the function $v(r,\theta,t) := u(r\cos\theta, r\sin\theta, t)$ satisfies

$$\frac{\partial v}{\partial r}(1,\theta,t) = 0, \quad v(r,\theta,0) = J_0(j_{1,1}\, r), \quad \text{and} \quad \frac{\partial v}{\partial t}(r,\theta,0) = -3J_0(j_{1,3}\, r)$$

for all $r \in [0,1]$, all $t \ge 0$ and all θ. Here $j_{1,1} < j_{1,2} < j_{1,3} < \cdots < j_{1,n} < \cdots$ are the positive zeros of the Bessel function $J_1(x)$ arranged in increasing order. (Recall that $\frac{d}{dx}J_0(x) = -J_1(x)$.)

7. The function $u(x,t)$ satisfies 1D heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x, \quad u_x(0,t) = 1, \quad u_x(1,t) = \alpha \text{ for all } x \in [0,1],\ \text{all } t \ge 0$$

(a) Find the constant α such that the equilibrium solution exists.

(b) Use the constant α in (a) and initial condition $u(x,0) = f(x)$ to find the solution $u(x,t)$.

8. Find solution to heat equation on the square $0 \le x \le L$, $0 \le y \le L$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2u$$

such that $u = 0$ on all four sides of the square and $u(x,y,0) = f(x,y)$. What is the condition on L so that every solution $u \to 0$ as $t \to \infty$.

9. Under what condition on $f(x)$, there is a solution to the Laplace equation on a rectangle $[0, L] \times [0, H]$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with boundary conditions

$$u_x(x, 0) = f(x),\ u_x(x, H) = 0,\ u_y(0, y) = 0,\ u_y(L, y) = 0$$

.

10. The displacement $u(r, \theta, t)$ of membrane of radius 1 with fixed boundary satisfies wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \quad 0 \le r \le 1, -\pi \le \theta \le \pi, t \ge 0$$

and boundary condition $u(1, \theta, t) = 0$. Find the solution $u(r, \theta, t)$ with initial conditions

$$u(r, \theta, 0) = J_0(z_{0,4}r),\ u_t(r, \theta, 0) = J_0(z_{0,2}r)$$